THE STANDARD AND ABRIDGED MOLDENSKY COORDINATE TRANSFORMATION FORMULAE

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ABSTRACT

The standard and abridged Molodensky coordinate transformation formulae are sometimes used by practitioners in the surveying and geodesy professions and are two of the standard models of coordinate transformation widely used in Geographic Information System (GIS) software. The formulae enable latitudes, longitudes and heights (ϕ, λ, h) related to one ellipsoid to be transformed to ϕ, λ, h related to another ellipsoid on the implicit assumptions that: (i) the X, Y, Z Cartesian axes of both ellipsoids are parallel, (ii) the coordinate differences $\delta X, \delta Y, \delta Z$ between the origins of the two reference ellipsoids are known and (iii) the defining geometric parameters of both reference ellipsoids are known. The formulae accept ϕ, λ, h as input variables and give changes $\delta\phi, \delta\lambda, \delta h$, thus the standard and abridged Molodensky transformation formulae are known as *curvilinear transformation formulae.* This paper provides a detailed derivation of the formulae together with a worked example that may be useful to the interested practitioner.

INTRODUCTION

In Molodensky *et al.* (1962), the authors derive a set of differential equations for transforming coordinates from one geodetic datum to another. Their equations (Molodensky *et al.,* (I.3.2), p. 14), linked changes in *x, y, z* Cartesian coordinates of a point with, (i) rotations $\varepsilon_x, \varepsilon_y, \varepsilon_z$ of the Cartesian axes about some fixed point x_0, y_0, z_0, (ii) "progressive translations" dx_0, dy_0, dz_0 of the ellipsoid origin between *x, y, z* Cartesian axes, and changes in the ellipsoid parameters δa and δf with changes in curvilinear coordinates $\delta\phi, \delta\lambda, \delta h$. Subsequent publications by other authors have described "Molodensky's" transformation in terms different from the original. This confusion was addressed by Soler (1976, p.2) who states:

> "… the differential equations published in the English translation of [Molodensky *et al.,* 1962] are equivalent to conventional conformal transformations. This dissipates the confusion created recently by some authors [Badekas, 1969], Krakiwsky and Thomson, 1974], who credited [Molodensky *et al.,* 1962] with a model they never wrote."

It is now common in the literature to describe three Molodensky transformations:

(i) The *Molodensky-Badekas* transformation: a seven-parameter conformal transformation (or similarity transformation) linking rotations $\varepsilon_X, \varepsilon_Y, \varepsilon_Z$ and translations $\delta X, \delta Y, \delta Z$ between the *X, Y, Z* Cartesian axes and a scale factor δs to changes in the Cartesian coordinates.

(ii) The *standard Molodensky* transformation: a five-parameter transformation linking translations $\delta X, \delta Y, \delta Z$ between the *X, Y, Z* Cartesian axes, and changes in the ellipsoid parameters δa and δf with changes in curvilinear coordinates $\delta\phi, \delta\lambda, \delta h$.

(iii) The *abridged Molodensky* transformation: a modified version of the standard Molodensky transformation obtained by certain simplifying assumptions. The abridged Molodensky transformation equations do not contain the ellipsoidal heights h of points to be transformed.

The standard and abridged Molodensky transformations, the subject of this paper, are two of the transformations adopted by the National Imagery and Mapping Agency (NIMA 2000) formerly the Defense Mapping Agency (DMA) and are also two of the methods recommended by the Geoscience Australia (ICSM 2003). It is the purpose of this paper is to set out a detailed derivation of the standard and abridged Molodensky transformations to provide the interested reader with some understanding of the mathematics involved. The derivation follows a method suggested by Krakiwsky and Wells (1971). A worked example is included.

DERIVATION OF THE STANDARD MOLODENSKY TRANSFORMATION FORMULAE

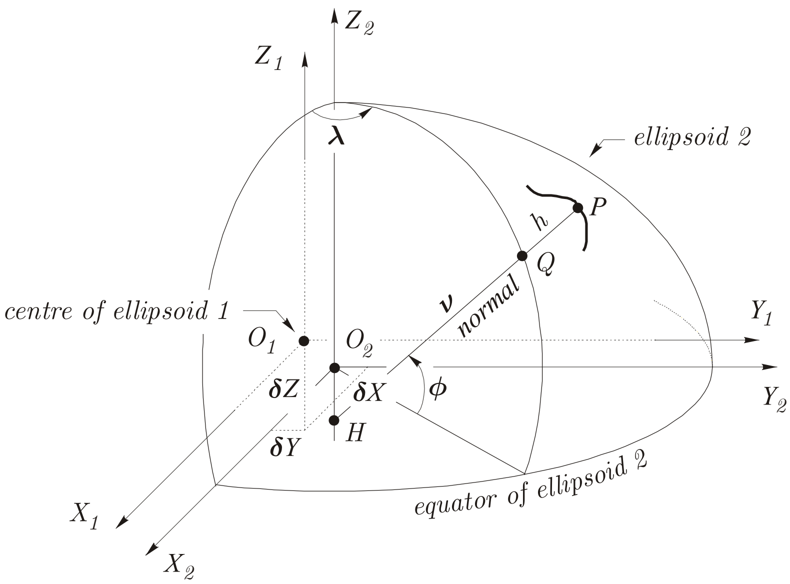

Figure 1 Sectional view of ellipsoid 1

Figure 1 shows a point $P(\phi, \lambda, h)$ on the Earth's terrestrial surface related to the centre O_2 of an ellipsoid (a_2, f_2) by the normal to the ellipsoid. The normal intersects the ellipsoidal surface at Q and the rotational axis of the ellipsoid at *H*. The distance PQ is the ellipsoidal height h and the distance $QH = \nu$ is the radius of curvature in the prime vertical plane of the ellipsoid. ϕ, λ, h are latitude, longitude and ellipsoidal height respectively, a is the semi-major axis of the ellipsoid and f is the flattening of the ellipsoid. O_1 is the centre of another ellipsoid (a_1, f_1) and two assumed parallel Cartesian coordinate axes are shown with origins at O_1 and O_2. The Z_1 and Z_2 axes are assumed to be parallel and are the rotational axes of the ellipsoids, the $X_1O_1Y_1$ and $X_2O_2Y_2$ equatorial planes are parallel and are the origin planes of latitude. The $X_1O_1Z_1$ and $X_2O_2Z_2$ planes are parallel and are the origin planes of longitude. The origins O_1 and O_2 are related by the translations $\delta X, \delta Y$ and δZ.

The Cartesian coordinates of a point, related to the centre of an ellipsoid, are

$$\begin{aligned} X &= (\nu + h)\cos\phi\cos\lambda \\ Y &= (\nu + h)\cos\phi\sin\lambda \\ Z &= \left(\nu\left(1 - e^2\right) + h\right)\sin\phi \end{aligned} \tag{1}$$

ν is the radius of curvature of the ellipsoid in the prime vertical plane and e^2 is the square of the eccentricity of the ellipsoid.

$$\begin{aligned} \nu &= \frac{a}{\left(1 - e^2\sin^2\phi\right)^{1/2}} \\ e^2 &= f(2 - f) \\ 1 - e^2 &= (1 - f)^2 \end{aligned} \tag{2a}$$

The eccentricity *e*, the flattening *f*, the semi-major and semi-minor axes of the ellipsoid (a and b, respectively) and ρ, the radius of curvature of the ellipsoid in the meridian plane are defined by the following equations

$$\begin{aligned} e^2 &= \frac{a^2 - b^2}{a^2} \\ f &= \frac{a - b}{a} \\ b &= a(1 - f) \\ \rho &= \frac{a\left(1 - e^2\right)}{\left(1 - e^2\sin^2\phi\right)^{3/2}} \end{aligned} \tag{2b}$$

Substituting (2a) into (1) gives the *X, Y, Z* coordinates as functions of *a, f,* ϕ, λ and *h.*

$$\begin{aligned} X &= \frac{a\cos\phi\cos\lambda}{\left(1 - f(2 - f)\sin^2\phi\right)^{1/2}} + h\cos\phi\cos\lambda \\ Y &= \frac{a\cos\phi\sin\lambda}{\left(1 - f(2 - f)\sin^2\phi\right)^{1/2}} + h\cos\phi\sin\lambda \\ Z &= \frac{a(1 - f)^2\sin\phi}{\left(1 - f(2 - f)\sin^2\phi\right)^{1/2}} + h\sin\phi \end{aligned} \tag{3}$$

Using the theorem of the *total differential* (Sokolnikoff & Redheffer 1966), small changes in the X, Y, Z coordinates can be linked to small changes in a, f, ϕ, λ and h.

$$
\begin{aligned}
\delta X &= \frac{\partial X}{\partial a}\delta a + \frac{\partial X}{\partial f}\delta f + \frac{\partial X}{\partial \phi}\delta \phi + \frac{\partial X}{\partial \lambda}\delta \lambda + \frac{\partial X}{\partial h}\delta h \\
\delta Y &= \frac{\partial Y}{\partial a}\delta a + \frac{\partial Y}{\partial f}\delta f + \frac{\partial Y}{\partial \phi}\delta \phi + \frac{\partial Y}{\partial \lambda}\delta \lambda + \frac{\partial Y}{\partial h}\delta h \\
\delta Z &= \frac{\partial Z}{\partial a}\delta a + \frac{\partial Z}{\partial f}\delta f + \frac{\partial Z}{\partial \phi}\delta \phi + \frac{\partial Z}{\partial \lambda}\delta \lambda + \frac{\partial Z}{\partial h}\delta h
\end{aligned}
\tag{4}
$$

In equations (4), all δ-values are formed by subtracting ellipsoid 1 values from ellipsoid 2 values, e.g., $\delta a = a_1 - a_2$. These equations are the basis of the standard Molodensky transformation.

The *standard Molodensky transformation formulae* are derived in the following manner.

The derivatives in (4) can be found from equations (3).

Derivatives $\dfrac{\partial X}{\partial a}, \dfrac{\partial Y}{\partial a}, \dfrac{\partial Z}{\partial a}$

$$
\frac{\partial X}{\partial a} = \frac{\cos\phi\cos\lambda}{\left(1 - f(2-f)\sin^2\phi\right)^{1/2}} = \frac{\cos\phi\cos\lambda}{\left(1 - e^2\sin^2\phi\right)^{1/2}} = \frac{\nu\cos\phi\cos\lambda}{a}
\tag{5a}
$$

Similarly

$$
\frac{\partial Y}{\partial a} = \frac{\nu\cos\phi\sin\lambda}{a}
\tag{5b}
$$

$$
\frac{\partial Z}{\partial a} = \frac{\nu(1-f)^2\sin\phi}{a} = \frac{\nu\left(1-e^2\right)\sin\phi}{a}
\tag{5c}
$$

Derivatives $\dfrac{\partial X}{\partial f}, \dfrac{\partial Y}{\partial f}, \dfrac{\partial Z}{\partial f}$

$$
\begin{aligned}
\frac{\partial X}{\partial f} &= -\frac{1}{2} a\cos\phi\cos\lambda\left(1 - f(2-f)\sin^2\phi\right)^{-3/2}\left(-(2-2f)\sin^2\phi\right) \\
&= \frac{a\cos\phi\cos\lambda(1-f)\sin^2\phi}{\left(1-e^2\sin^2\phi\right)^{3/2}}
\end{aligned}
$$

but $\rho = \dfrac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{3/2}} = \dfrac{a(1-f)^2}{\left(1-e^2\sin^2\phi\right)^{3/2}}$ is the radius of curvature in the meridian plane,

hence

$$
\frac{\partial X}{\partial f} = \frac{\rho\sin^2\phi}{1-f}\cos\phi\cos\lambda
\tag{6a}
$$

Similarly

$$
\frac{\partial Y}{\partial f} = \frac{\rho\sin^2\phi}{1-f}\cos\phi\sin\lambda
\tag{6b}
$$

Using the quotient rule of calculus: $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$$
\begin{aligned}
\frac{\partial Z}{\partial f} &= \frac{\left(1-f(2-f)\sin^2\phi\right)^{1/2}\left(-2a\sin\phi(1-f)\right) - a(1-f)^2\sin\phi\left(-\left(1-f(2-f)\sin^2\phi\right)^{-1/2}(1-f)\sin^2\phi\right)}{1-f(2-f)\sin^2\phi} \\
&= \frac{-2a\sin\phi(1-f)\left(1-e^2\sin^2\phi\right)^{1/2} + a\left(1-e^2\right)\sin\phi\left(1-e^2\sin^2\phi\right)^{-1/2}(1-f)\sin^2\phi}{1-e^2\sin^2\phi} \\
&= \frac{-2a\sin\phi(1-f)}{\left(1-e^2\sin^2\phi\right)^{1/2}} + \frac{a\left(1-e^2\right)\sin^2\phi\sin\phi(1-f)}{\left(1-e^2\sin^2\phi\right)^{3/2}} \\
&= -2\nu\sin\phi(1-f) + \rho\sin^2\phi(1-f)
\end{aligned}
$$

giving

$$
\frac{\partial Z}{\partial f} = \left(\rho\sin^2\phi - 2\nu\right)\sin\phi(1-f) \tag{6c}
$$

Derivatives $\dfrac{\partial X}{\partial \phi}, \dfrac{\partial Y}{\partial \phi}, \dfrac{\partial Z}{\partial \phi}$

$$
\begin{aligned}
\frac{\partial X}{\partial \phi} &= \frac{\left(1-e^2\sin^2\phi\right)^{1/2}\left(-a\sin\phi\cos\lambda\right) - a\cos\phi\cos\lambda\left(-\left(1-e^2\sin^2\phi\right)^{-1/2}e^2\sin\phi\cos\phi\right)}{1-e^2\sin^2\phi} \\
&\quad -h\sin\phi\cos\lambda \\
&= \frac{-a\sin\phi\cos\lambda}{\left(1-e^2\sin^2\phi\right)^{1/2}} + \frac{a\cos\phi\cos\lambda\, e^2\sin\phi\cos\phi}{\left(1-e^2\sin^2\phi\right)^{3/2}} - h\sin\phi\cos\lambda \\
&= -\nu\sin\phi\cos\lambda + \nu\sin\phi\cos\lambda\frac{e^2\cos^2\phi}{1-e^2\sin^2\phi} - h\sin\phi\cos\lambda \\
&= -\nu\sin\phi\cos\lambda\left\{1-\frac{e^2\cos^2\phi}{1-e^2\sin^2\phi}\right\} - h\sin\phi\cos\lambda
\end{aligned}
$$

The term in braces { } is

$$
\{\ \} = \frac{1-e^2\sin^2\phi - e^2\cos^2\phi}{1-e^2\sin^2\phi} = \frac{1-e^2\left(\sin^2\phi+\cos^2\phi\right)}{1-e^2\sin^2\phi} = \frac{1-e^2}{1-e^2\sin^2\phi} = \frac{\rho}{\nu}
$$

giving

$$
\frac{\partial X}{\partial \phi} = -(\rho+h)\sin\phi\cos\lambda \tag{7a}
$$

Similarly

$$
\frac{\partial Y}{\partial \phi} = -(\rho+h)\sin\phi\sin\lambda \tag{7b}
$$

$$
\begin{aligned}
\frac{\partial Z}{\partial \phi} &= \frac{\left(1-e^2\sin^2\phi\right)^{1/2}a\left(1-e^2\right)\cos\phi - a\left(1-e^2\right)\sin\phi\left(\left(1-e^2\sin^2\phi\right)^{-1/2}e^2\sin\phi\cos\phi\right)}{1-e^2\sin^2\phi} \\
&\quad +h\cos\phi \\
&= \frac{a\left(1-e^2\right)\cos\phi}{\left(1-e^2\sin^2\phi\right)^{1/2}}\frac{1-e^2\sin^2\phi}{1-e^2\sin^2\phi} + \frac{a\left(1-e^2\right)\cos\phi}{\left(1-e^2\sin^2\phi\right)^{3/2}}e^2\sin^2\phi + h\cos\phi \\
&= \rho\cos\phi\left(1-e^2\sin^2\phi\right) + \rho\cos\phi\, e^2\sin^2\phi + h\cos\phi
\end{aligned}
$$

giving

$$\frac{\partial Z}{\partial \phi} = (\rho + h)\cos\phi \tag{7c}$$

Derivatives $\dfrac{\partial X}{\partial \lambda}, \dfrac{\partial Y}{\partial \lambda}, \dfrac{\partial Z}{\partial \lambda}$

$$\begin{aligned}\frac{\partial X}{\partial \lambda} &= \frac{\left(1 - e^2 \sin^2\phi\right)^{1/2}(-a\cos\phi\sin\lambda) - 0}{1 - e^2\sin^2\phi} - h\cos\phi\sin\lambda \\ &= \frac{-a}{\left(1 - e^2\sin^2\phi\right)^{1/2}}\cos\phi\sin\lambda - h\cos\phi\sin\lambda\end{aligned}$$

giving

$$\frac{\partial X}{\partial \lambda} = -(\nu + h)\cos\phi\sin\lambda \tag{8a}$$

Similarly

$$\frac{\partial Y}{\partial \lambda} = (\nu + h)\cos\phi\cos\lambda \tag{8b}$$

$$\frac{\partial Z}{\partial \lambda} = 0 \tag{8c}$$

Derivatives $\dfrac{\partial X}{\partial h}, \dfrac{\partial Y}{\partial h}, \dfrac{\partial Z}{\partial h}$

$$\frac{\partial X}{\partial h} = \cos\phi\cos\lambda \tag{9a}$$

$$\frac{\partial Y}{\partial h} = \cos\phi\sin\lambda \tag{9b}$$

$$\frac{\partial Z}{\partial h} = \sin\phi \tag{9c}$$

Equations (4) can be expressed in matrix form

$$\begin{bmatrix}\delta X \\ \delta Y \\ \delta Z\end{bmatrix} = \mathbf{B}\begin{bmatrix}\delta a \\ \delta f\end{bmatrix} + \mathbf{J}\begin{bmatrix}\delta\phi \\ \delta\lambda \\ \delta h\end{bmatrix} \tag{10}$$

where

$$\mathbf{B} = \begin{bmatrix}\dfrac{\partial X}{\partial a} & \dfrac{\partial X}{\partial f} \\ \dfrac{\partial Y}{\partial a} & \dfrac{\partial Y}{\partial f} \\ \dfrac{\partial Z}{\partial a} & \dfrac{\partial Z}{\partial f}\end{bmatrix} = \begin{bmatrix}\dfrac{\nu\cos\phi\cos\lambda}{a} & \dfrac{\rho\sin^2\phi}{1-f}\cos\phi\cos\lambda \\ \dfrac{\nu\cos\phi\sin\lambda}{a} & \dfrac{\rho\sin^2\phi}{1-f}\cos\phi\sin\lambda \\ \dfrac{\nu(1-f)^2\sin\phi}{a} & \left(\rho\sin^2\phi - 2\nu\right)\sin\phi(1-f)\end{bmatrix} \tag{11}$$

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial X}{\partial \phi} & \dfrac{\partial X}{\partial \lambda} & \dfrac{\partial X}{\partial h} \\ \dfrac{\partial Y}{\partial \phi} & \dfrac{\partial Y}{\partial \lambda} & \dfrac{\partial Y}{\partial h} \\ \dfrac{\partial Z}{\partial \phi} & \dfrac{\partial Z}{\partial \lambda} & \dfrac{\partial Z}{\partial h} \end{bmatrix} = \begin{bmatrix} -(\rho+h)\sin\phi\cos\lambda & -(\nu+h)\cos\phi\sin\lambda & \cos\phi\cos\lambda \\ -(\rho+h)\sin\phi\sin\lambda & (\nu+h)\cos\phi\cos\lambda & \cos\phi\sin\lambda \\ (\rho+h)\cos\phi & 0 & \sin\phi \end{bmatrix} \tag{12}$$

The small changes in ellipsoidal coordinates $\delta\phi, \delta\lambda, \delta h$ can be obtained from (10)

$$\begin{bmatrix} \delta\phi \\ \delta\lambda \\ \delta h \end{bmatrix} = -\mathbf{J}^{-1} \left\{ \begin{bmatrix} -\delta X \\ -\delta Y \\ -\delta Z \end{bmatrix} + \mathbf{B} \begin{bmatrix} \delta a \\ \delta f \end{bmatrix} \right\} \tag{13}$$

The inverse of **J** can be found by the method of *cofactors* and *adjoints* (Mikhail 1973, pp. 442-5).

For a 3×3 matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ the inverse $\mathbf{A}^{-1}$ is given by

$$\mathbf{A}^{-1} = \frac{adj\,\mathbf{A}}{|\mathbf{A}|}$$

where $adj\,\mathbf{A}$ is the *adjoint* matrix and $|\mathbf{A}|$ is the *determinant* of **A**, a scalar quantity. Each element a_{ij} of **A** has a *minor* m_{ij} and a *cofactor* c_{ij}. The minor of each element is the determinant of the elements of **A** remaining after row i and column j are deleted, eg, $m_{11} = a_{22}a_{33} - a_{23}a_{32}$, $m_{22} = a_{11}a_{33} - a_{13}a_{31}$ and $m_{32} = a_{11}a_{23} - a_{13}a_{21}$. The cofactors $c_{ij} = (-1)^{i+j}\, m_{ij}$ form a matrix **C** whose transpose is the adjoint matrix..

The determinant $|\mathbf{A}| = \sum_{j=1}^{3} a_{ij} c_{ij}$.

The elements of the cofactor matrix of **J** are

$$\begin{aligned} c_{11} &= +\left\{[(\nu+h)\cos\phi\cos\lambda][\sin\phi] - [\cos\phi\sin\lambda][0]\right\} \\ &= (\nu+h)\sin\phi\cos\phi\cos\lambda \end{aligned}$$

$$\begin{aligned} c_{12} &= -\left\{[-(\rho+h)\sin\phi\sin\lambda][\sin\phi] - [\cos\phi\sin\lambda][(\rho+h)\cos\phi]\right\} \\ &= -\left\{-(\rho+h)\sin\lambda\left(\sin^2\phi + \cos^2\phi\right)\right\} \\ &= (\rho+h)\sin\lambda \end{aligned}$$

$$\begin{aligned} c_{13} &= +\left\{[-(\rho+h)\sin\phi\sin\lambda][0] - [(\nu+h)\cos\phi\cos\lambda][(\rho+h)\cos\phi]\right\} \\ &= -(\nu+h)\cos\phi\cos\lambda(\rho+h)\cos\phi \end{aligned}$$

$$\begin{aligned} c_{21} &= -\left\{[-(\nu+h)\cos\phi\sin\lambda][\sin\phi] - [\cos\phi\cos\lambda][0]\right\} \\ &= (\nu+h)\sin\phi\cos\phi\sin\lambda \end{aligned}$$

$$
\begin{aligned}
c_{22} &= +\left\{\left[-(\rho+h)\sin\phi\cos\lambda\right]\left[\sin\phi\right]-\left[\cos\phi\cos\lambda\right]\left[(\rho+h)\cos\phi\right]\right\}\\
&= -(\rho+h)\cos\lambda\left(\sin^2\phi+\cos^2\phi\right)\\
&= -(\rho+h)\cos\lambda
\end{aligned}
$$

$$
\begin{aligned}
c_{23} &= -\left\{\left[-(\rho+h)\sin\phi\cos\lambda\right]\left[0\right]-\left[-(\nu+h)\cos\phi\sin\lambda\right]\left[(\rho+h)\cos\phi\right]\right\}\\
&= -(\nu+h)\cos\phi\sin\lambda(\rho+h)\cos\phi
\end{aligned}
$$

$$
\begin{aligned}
c_{31} &= +\left\{\left[-(\nu+h)\cos\phi\sin\lambda\right]\left[\cos\phi\sin\lambda\right]-\left[\cos\phi\cos\lambda\right]\left[(\nu+h)\cos\phi\cos\lambda\right]\right\}\\
&= -(\nu+h)\cos^2\phi\left(\sin^2\lambda+\cos^2\lambda\right)\\
&= -(\nu+h)\cos^2\phi
\end{aligned}
$$

$$
\begin{aligned}
c_{32} &= -\left\{\left[-(\rho+h)\sin\phi\cos\lambda\right]\left[\cos\phi\sin\lambda\right]-\left[\cos\phi\cos\lambda\right]\left[-(\rho+h)\sin\phi\sin\lambda\right]\right\}\\
&= -\left\{-(\rho+h)\cos\phi\cos\lambda(\sin\phi\sin\lambda-\sin\phi\sin\lambda)\right\}\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
c_{33} &= +\left\{\left[-(\rho+h)\sin\phi\cos\lambda\right]\left[(\nu+h)\cos\phi\cos\lambda\right]-\left[-(\nu+h)\cos\phi\sin\lambda\right]\left[-(\rho+h)\sin\phi\sin\lambda\right]\right\}\\
&= +\left\{-(\rho+h)\sin\phi\cos\phi(\nu+h)\left(\cos^2\lambda+\sin^2\lambda\right)\right\}\\
&= -(\rho+h)\sin\phi(\nu+h)\cos\phi
\end{aligned}
$$

The determinant $|\mathbf{J}|$ is given by

$$
\begin{aligned}
|\mathbf{J}| &= j_{31}c_{31}+j_{32}c_{32}+j_{33}c_{33}\\
&= (\rho+h)\cos\phi\left(-(\nu+h)\cos^2\phi\right)+0+\sin\phi\left(-(\rho+h)\sin\phi(\nu+h)\cos\phi\right)\\
&= -(\nu+h)\cos\phi\left((\rho+h)\cos^2\phi+(\rho+h)\sin^2\phi\right)\\
&= -(\nu+h)\cos\phi(\rho+h)
\end{aligned}
$$

The inverse $\mathbf{J}^{-1}=\dfrac{adj\,\mathbf{J}}{|\mathbf{J}|}=\dfrac{\mathbf{C}^T}{|\mathbf{J}|}$

$$
\mathbf{J}^{-1}=\begin{bmatrix}
\dfrac{-\sin\phi\cos\lambda}{\rho+h} & \dfrac{-\sin\phi\sin\lambda}{\rho+h} & \dfrac{\cos\phi}{\rho+h}\\
\dfrac{-\sin\lambda}{(\nu+h)\cos\phi} & \dfrac{\cos\lambda}{(\nu+h)\cos\phi} & 0\\
\cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi
\end{bmatrix}
\tag{14}
$$

Expanding (13) and substituting $\mathbf{J}^{-1}$ gives

$$
\begin{bmatrix}\delta\phi\\ \delta\lambda\\ \delta h\end{bmatrix}=
\begin{bmatrix}
\dfrac{-\sin\phi\cos\lambda}{\rho+h} & \dfrac{-\sin\phi\sin\lambda}{\rho+h} & \dfrac{\cos\phi}{\rho+h}\\
\dfrac{-\sin\lambda}{(\nu+h)\cos\phi} & \dfrac{\cos\lambda}{(\nu+h)\cos\phi} & 0\\
\cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi
\end{bmatrix}
\begin{bmatrix}
\dfrac{\nu\cos\phi\cos\lambda}{a}\delta a+\dfrac{\rho\sin^2\phi\cos\phi\cos\lambda}{1-f}\delta f-\delta X\\
\dfrac{\nu\cos\phi\sin\lambda}{a}\delta a+\dfrac{\rho\sin^2\phi\cos\phi\sin\lambda}{1-f}\delta f-\delta Y\\
\dfrac{\nu(1-f)^2\sin\phi}{a}\delta a+\left(\rho\sin^2\phi-2\nu\right)\sin\phi(1-f)\delta f-\delta Z
\end{bmatrix}
\tag{15}
$$

Multiplying the right-hand-side of (15) and simplifying gives an expression for $\delta\phi$

$$\begin{aligned}\delta\phi = \frac{1}{\rho+h}\{&-\delta X\sin\phi\cos\lambda-\delta Y\sin\phi\sin\lambda+\delta Z\cos\phi\\&+\frac{\nu e^2\sin\phi\cos\phi}{a}\delta a\\&+\frac{\sin\phi\cos\phi}{1-f}\left(\rho e^2\sin^2\phi+2\nu\left(1-e^2\right)\right)\delta f\Big\}\end{aligned}\tag{16}$$

The third term in the braces { } on the right-hand-side of (16) can be simplified in the following manner

$$\begin{aligned}3^{\text{rd}}\ \text{term} &= \frac{\sin\phi\cos\phi}{1-f}\left(\rho e^2\sin^2\phi+\rho-\rho+2\nu\left(1-e^2\right)\right)\\&=\frac{\sin\phi\cos\phi}{1-f}\left(-\rho\left(1-e^2\sin^2\phi\right)+\rho+2\nu\left(1-e^2\right)\right)\end{aligned}$$

but $\dfrac{\rho}{\nu}=\dfrac{1-e^2}{1-e^2\sin^2\phi}$ giving $-\rho\left(1-e^2\sin^2\phi\right)=-\nu\left(1-e^2\right)$ hence

$$3^{\text{rd}}\ \text{term} = \frac{\sin\phi\cos\phi}{1-f}\left(\rho+\nu\left(1-e^2\right)\right)=\sin\phi\cos\phi\left(\frac{\rho}{1-f}+\nu(1-f)\right)$$

Substituting into (16) gives

$$\boxed{\begin{aligned}\delta\phi = \frac{1}{\rho+h}\{&-\delta X\sin\phi\cos\lambda-\delta Y\sin\phi\sin\lambda+\delta Z\cos\phi\\&+\frac{\nu e^2\sin\phi\cos\phi}{a}\delta a\\&+\sin\phi\cos\phi\left(\frac{\rho}{1-f}+\nu(1-f)\right)\delta f\Big\}\end{aligned}}\tag{17}$$

Multiplying the right-hand-side of (15) and simplifying gives and expression for $\delta\lambda$

$$\boxed{\delta\lambda=\frac{1}{(\nu+h)\cos\phi}(-\delta X\sin\lambda+\delta Y\cos\lambda)}\tag{18}$$

Multiplying the right-hand-side of (15) and simplifying gives and expression for δh

$$\begin{aligned}\delta h = {}&\delta X\cos\phi\cos\lambda+\delta Y\cos\phi\sin\lambda+\delta Z\sin\phi\\&+\left\{-\frac{\nu}{a}+\frac{\nu e^2\sin^2\phi}{a}\right\}\delta a\\&+\left\{\frac{-\rho\sin^2\phi\cos^2\phi}{1-f}-\left(\rho\sin^2\phi-2\nu\right)\sin^2\phi(1-f)\right\}\delta f\end{aligned}\tag{19}$$

The second term in the braces { } on the right-hand-side of (19) can be simplified in the following manner

$$2^{\text{nd}}\ \text{term} = \frac{-\nu}{a}\left(1-e^2\sin^2\phi\right)$$

and since $\nu=\dfrac{a}{\left(1-e^2\sin^2\phi\right)^{1/2}}$ then $1-e^2\sin^2\phi=\dfrac{a^2}{\nu^2}$ and

$$2^{\text{nd}}\ \text{term} = \frac{-a}{\nu}$$

The third term in the braces { } on the right-hand-side of (19) can be simplified in the following manner

$$\begin{aligned}
3^{\text{rd}}\ \text{term} &= \frac{\sin^2\phi}{1-f}\left(-\rho\cos^2\phi - \rho\sin^2\phi\,(1-f)^2 + 2\nu\,(1-f)^2\right) \\
&= \frac{\sin^2\phi}{1-f}\left(-\rho\cos^2\phi - \rho\sin^2\phi + \rho e^2\sin^2\phi + 2\nu\left(1-e^2\right)\right) \\
&= \frac{\sin^2\phi}{1-f}\left(-\rho\left(1-e^2\sin^2\phi\right) + 2\nu\left(1-e^2\right)\right) \\
&= \frac{\sin^2\phi}{1-f}\left(-\nu\left(1-e^2\right) + 2\nu\left(1-e^2\right)\right) \\
&= \frac{\sin^2\phi}{1-f}\,\nu\left(1-e^2\right)
\end{aligned}$$

Now, since $1-e^2 = (1-f)^2$

$$3^{\text{rd}}\ \text{term} = \nu\,(1-f)\sin^2\phi$$

Substituting these expressions into (19) gives

$$\boxed{\delta h = \delta X\cos\phi\cos\lambda + \delta Y\cos\phi\sin\lambda + \delta Z\sin\phi - \frac{a}{\nu}\delta a + \nu\,(1-f)\sin^2\phi\,\delta f} \tag{20}$$

Equations (17), (18) and (20) are the *standard Molodensky transformation formulae*

$$\boxed{\begin{aligned}
\delta\phi &= \frac{1}{\rho+h}\Big\{-\delta X\sin\phi\cos\lambda - \delta Y\sin\phi\sin\lambda + \delta Z\cos\phi \\
&\qquad + \frac{\nu e^2\sin\phi\cos\phi}{a}\delta a \\
&\qquad + \sin\phi\cos\phi\left(\frac{\rho}{1-f} + \nu\,(1-f)\right)\delta f\Big\} \\
\delta\lambda &= \frac{1}{(\nu+h)\cos\phi}\left(-\delta X\sin\lambda + \delta Y\cos\lambda\right) \\
\delta h &= \delta X\cos\phi\cos\lambda + \delta Y\cos\phi\sin\lambda + \delta Z\sin\phi - \frac{a}{\nu}\delta a + \nu\,(1-f)\sin^2\phi\,\delta f
\end{aligned}} \tag{21}$$

They can be combined and represented as a matrix equation

$$\begin{bmatrix}\delta\phi\\ \delta\lambda\\ \delta h\end{bmatrix} = \begin{bmatrix}
\dfrac{-\sin\phi\cos\lambda}{\rho+h} & \dfrac{-\sin\phi\sin\lambda}{\rho+h} & \dfrac{\cos\phi}{\rho+h} & \dfrac{\nu e^2\sin\phi\cos\phi}{a(\rho+h)} & \dfrac{\sin\phi\cos\phi}{\rho+h}\left(\dfrac{\rho}{1-f} + \nu\,(1-f)\right) \\
\dfrac{-\sin\lambda}{(\nu+h)\cos\phi} & \dfrac{\cos\lambda}{(\nu+h)\cos\phi} & 0 & 0 & 0 \\
\cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi & \dfrac{-a}{\nu} & \nu\,(1-f)\sin^2\phi
\end{bmatrix}\begin{bmatrix}\delta X\\ \delta Y\\ \delta Z\\ \delta a\\ \delta f\end{bmatrix} \tag{22}$$

It is common in the literature to denote the radii of curvature of the ellipsoid as

$$R_N = \upsilon = \text{ radius of curvature in the prime vertical plane}$$
$$R_M = \rho = \text{ radius of curvature in the meridian plane}$$

Noting that $b = a(1-f)$, $\dfrac{b}{a} = (1-f)$, $\dfrac{a}{b} = \dfrac{1}{(1-f)}$ an alternative presentation of the *standard Molodensky transformation formulae* is

$$\boxed{\begin{aligned}
\delta\phi &= \frac{1}{R_M + h}\Big\{-\delta X \sin\phi\cos\lambda - \delta Y \sin\phi\sin\lambda + \delta Z\cos\phi \\
&\qquad + \frac{R_N\, e^2 \sin\phi\cos\phi}{a}\delta a \\
&\qquad + \sin\phi\cos\phi\left(R_M\,\frac{a}{b} + R_N\,\frac{b}{a}\right)\delta f\Big\} \\
\\
\delta\lambda &= \frac{1}{(R_N + h)\cos\phi}(-\delta X\sin\lambda + \delta Y\cos\lambda) \\
\\
\delta h &= \delta X\cos\phi\cos\lambda + \delta Y\cos\phi\sin\lambda + \delta Z\sin\phi - \frac{a}{R_N}\delta a + R_N\,\frac{b}{a}\sin^2\phi\,\delta f
\end{aligned}} \tag{23}$$

DERIVATION OF THE ABRIDGED MOLODENSKY TRANSFORMATION FORMULAE

The *abridged Molodensky transformation formulae* do not contain the parameter *h,* the ellipsoidal heights of the points to be transformed. Their derivation depends upon several approximations that are set out below.

Approximations involving ν: the radius of curvature in the prime vertical plane

$$\nu = \frac{a}{\left(1 - e^2\sin^2\phi\right)^{1/2}} \tag{24a}$$

The denominator of (24a) can be simplified by using the *Binomial Theorem*

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

giving

$$\begin{aligned}
\left(1 - e^2\sin^2\phi\right)^{-1/2} &= 1 + \left(-\frac{1}{2}\right)\left(-e^2\sin^2\phi\right) + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-e^2\sin^2\phi\right)^2 + \cdots \\
&= 1 + \frac{1}{2}e^2\sin^2\phi + \frac{3}{8}e^4\sin^4\phi + \cdots
\end{aligned}$$

Now $e^2 = f(2-f) = 2f - f^2$ and substituting into the equation above gives

$$\left(1-e^2\sin^2\phi\right)^{-1/2} = 1+\frac{1}{2}\left(2f-f^2\right)\sin^2\phi+\frac{3}{8}\left(2f-f^2\right)^2\sin^4\phi+\cdots$$

The flattening f is a small quantity $(f \approx 0.003)$ and f^2 is exceedingly small $\left(f^2 \approx 0.00001\right)$; hence, in the equation above, ignoring terms containing $f^2, f^3, f^4,$ etc

$$\left(1-e^2\sin^2\phi\right)^{-1/2} \approx 1+f\sin^2\phi \tag{24b}$$

Substituting (24b) into (24a) gives

$$\frac{\nu}{a} \approx 1+f\sin^2\phi \tag{24c}$$

Other approximations may be derived from (24c), again ignoring terms $f^2, f^3, f^4,$ etc

$$\begin{aligned}\frac{\nu e^2}{2a} &\approx \frac{1}{2}e^2\left(1+f\sin^2\phi\right)\\ &= \frac{1}{2}\left(2f-f^2\right)\left(1+f\sin^2\phi\right)\\ &= \frac{1}{2}\left(2f+2f^2\sin^2\phi-f^2-f^3\sin^2\phi\right)\\ &\approx f\end{aligned} \tag{24d}$$

$$\begin{aligned}\nu(1-f) &\approx a(1-f)\left(1+f\sin^2\phi\right)\\ &= a-af+af\sin^2\phi-af^2\sin^2\phi\\ &\approx a-af+af\sin^2\phi\end{aligned} \tag{24e}$$

Approximations involving ρ: the radius of curvature in the meridian plane

$$\rho = \frac{a\left(1-e^2\right)}{\left(1-e^2\sin^2\phi\right)^{3/2}} \tag{25a}$$

The denominator of (25a) can be simplified by using the *Binomial Theorem*

$$\begin{aligned}\left(1-e^2\sin^2\phi\right)^{-3/2} &= 1+\left(-\frac{3}{2}\right)\left(-e^2\sin^2\phi\right)+\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-e^2\sin^2\phi\right)^2+\cdots\\ &= 1+\frac{3}{2}e^2\sin^2\phi+\frac{15}{8}e^4\sin^4\phi+\cdots\end{aligned}$$

Now $e^2 = f(2-f) = 2f-f^2$ and substituting into the equation above gives

$$\left(1-e^2\sin^2\phi\right)^{-3/2} = 1+\frac{3}{2}\left(2f-f^2\right)\sin^2\phi+\frac{15}{8}\left(2f-f^2\right)^2\sin^4\phi+\cdots$$

Ignoring terms containing $f^2, f^3, f^4,$ etc

$$\left(1-e^2\sin^2\phi\right)^{-3/2} \approx 1+3f\sin^2\phi \tag{25b}$$

Substituting (25b) into (25a) gives

$$\frac{\rho}{a\left(1-e^2\right)} \approx 1+3f\sin^2\phi \tag{25c}$$

Now $1 - e^2 = (1-f)^2$ and (25c) may be re-arranged as

$$\begin{aligned} \frac{\rho}{(1-f)} &\approx a(1-f)\left(1+3f\sin^2\phi\right) \\ &= a + 3af\sin^2\phi - af - 3af^2\sin^2\phi \\ &\approx a - af + 3af\sin^2\phi \end{aligned} \tag{25d}$$

Approximations given in equations (24) and (25) are used in the following derivation of the *abridged Molodensky transformation formulae*

Equation for $\delta\phi$:

The 2nd and 3rd lines of the equation for $\delta\phi$ in (21) can be combined as

$$2\sin\phi\cos\phi\left\{\left(\frac{\nu e^2}{2a}\right)\delta a + \left(\frac{\rho}{2(1-f)} + \frac{\nu(1-f)}{2}\right)\delta f\right\} \tag{26}$$

Using the approximation in (24d) the first term in the braces { } in (26) can be written as

$$\left(\frac{\nu e^2}{2a}\right)\delta a \approx f\,\delta a \tag{27a}$$

Using the approximations in (24e) and (25d) the second term in the braces { } in (26) can be written as

$$\begin{aligned} \left(\frac{\rho}{2(1-f)} + \frac{\nu(1-f)}{2}\right)\delta f &\approx \frac{1}{2}\left(a - af + 3af\sin^2\phi + a - af + af\sin^2\phi\right)\delta f \\ &= \left(a - af + 2af\sin^2\phi\right)\delta f \end{aligned} \tag{27b}$$

f is a small quantity $(f \approx 0.003)$ and $\delta f = f_1 - f_2$, the difference in ellipsoid flattening, will be very small $\left(\delta f \approx 8\times 10^{-8}\right)$ and products $f\delta f \approx 0$. Hence, (27b) can be written as

$$\left(\frac{\rho}{2(1-f)} + \frac{\nu(1-f)}{2}\right)\delta f \approx a\,\delta f \tag{27c}$$

Substituting (27a) and (27c) into equation (26) and noting that $2\sin\phi\cos\phi = \sin 2\phi$ we have an approximation

$$2\sin\phi\cos\phi\left\{\left(\frac{\nu e^2}{2a}\right)\delta a + \left(\frac{\rho}{2(1-f)} + \frac{\nu(1-f)}{2}\right)\delta f\right\} \approx \left(f\,\delta a + a\,\delta f\right)\sin 2\phi$$

Using this approximation and ignoring the ellipsoidal height *h,* we may write the equation for $\delta\phi$ as

$$\delta\phi = \frac{1}{\rho}\left\{-\delta X\sin\phi\cos\lambda - \delta Y\sin\phi\sin\lambda + \delta Z\cos\phi + \left(f\,\delta a + a\,\delta f\right)\sin 2\phi\right\} \tag{28}$$

Equation for $\delta\lambda$:

The equation for $\delta\lambda$ in (21) is modified by omitting the parameter h giving

$$\delta\lambda = \frac{1}{\nu\cos\phi}(-\delta X\sin\lambda + \delta Y\cos\lambda) \tag{29}$$

Equation for δh :

In the equation for δh in (21) the last two terms can be written as

$$A = -\frac{a}{\nu}\delta a + \nu(1-f)\sin^2\phi\,\delta f \tag{30a}$$

Using the approximation in equation (24c) we may write

$$\frac{a}{\nu} \approx \left(1 + f\sin^2\phi\right)^{-1}$$

Expanding this approximation using the Binomial Theorem gives

$$\begin{aligned}\frac{a}{\nu} &= 1 + (-1)\left(f\sin^2\phi\right) + \frac{1}{2}(-1)(-2)\left(f\sin^2\phi\right)^2 + \cdots \\ &= 1 - f\sin^2\phi + f^2\sin^4\phi + \cdots \\ &\approx 1 - f\sin^2\phi \end{aligned} \tag{30b}$$

Using the approximation in equation (24e) and noting that products $f\delta f \approx 0$

$$\begin{aligned}\nu(1-f)\sin^2\phi\,\delta f &\approx \left(a - af + af\sin^2\phi\right)\sin^2\phi\,\delta f \\ &= af\sin^2\phi - af\delta f\sin^2\phi + af\delta f\sin^4\phi \\ &\approx a\,\delta f\sin^2\phi \end{aligned} \tag{30c}$$

Using approximations (30b) and (30c), equation (30a) can be written as

$$A = -\frac{a}{\nu}\delta a + \nu(1-f)\sin^2\phi\,\delta f \approx -\delta a + \left(f\,\delta a + a\,\delta f\right)\sin^2\phi \tag{30d}$$

Using (30d), the equation for δh in (21) can be written as

$$\delta h = \delta X\cos\phi\cos\lambda + \delta Y\cos\phi\sin\lambda + \delta Z\sin\phi - \delta a + \left(f\,\delta a + a\,\delta f\right)\sin^2\phi \tag{31}$$

Equations (28), (29) and (31) are the *abridged Molodensky transformation formulae*

$$\boxed{\begin{aligned}\delta\phi &= \frac{1}{\rho}\left\{-\delta X\sin\phi\cos\lambda - \delta Y\sin\phi\sin\lambda + \delta Z\cos\phi + \left(f\,\delta a + a\,\delta f\right)\sin 2\phi\right\} \\ \delta\lambda &= \frac{1}{\nu\cos\phi}(-\delta X\sin\lambda + \delta Y\cos\lambda) \\ \delta h &= \delta X\cos\phi\cos\lambda + \delta Y\cos\phi\sin\lambda + \delta Z\sin\phi - \delta a + \left(f\,\delta a + a\,\delta f\right)\sin^2\phi \end{aligned}} \tag{32}$$

COMMENT ON THE DERIVATIONS

The derivation of the standard Molodensky transformation formulae follows a method suggested by Krakiwsky and Wells (1971). The author has not found a derivation of these formulae in the readily available Geodesy and Surveying textbooks although it surely exists in the associated literature (technical reports, papers and associated articles) and the author makes no claims of originality. The abridged Molodensky transformation formulae that were probably derived in an era when formulae were "difficult" to evaluate and simplifications were warranted no longer seem to have the relevance they once might have had. Simply setting the ellipsoidal height h to zero in the standard Molodensky transformation formulae achieves the same result.

A worked example is set out in detail in the Appendices and may be useful in checking computer programs. Test values are computed that can be used to verify the transformation.

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APPENDIX 1

TEST VALUES FOR MOLODENSKY TRANSFORMATION

Transformation from Australian Geodetic Datum 1966 (AGD66) to World Geodetic System 1984 (WGS84).

AGD66 Geodetic coordinates: $\phi = -37^\circ\, 48'\, 00.0000''$ $\lambda = +144^\circ\, 58'\, 00.0000''$ $h = 50.000$ m

AGD66 Ellipsoid parameters: $a = 6378160$ m $f = 1/298.25$

AGD66 Cartesian coordinates:

$$\begin{aligned}
X &= (\nu + h)\cos\phi\cos\lambda &&= -4131857.9379 \text{ m} \\
Y &= (\nu + h)\cos\phi\sin\lambda &&= +2896741.9218 \text{ m} \\
Z &= \left(\nu\left(1-e^2\right)+h\right)\sin\phi &&= -3887971.3157 \text{ m} \\
e^2 &= f\,(2-f) &&= 6.69454185459\times 10^{-3} \\
\nu &= \frac{a}{\left(1-e^2\sin^2\phi\right)^{1/2}} &&= 6386195.1797 \text{ m}
\end{aligned}$$

WGS84 Cartesian coordinates:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{WGS84} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{AGD66} + \begin{bmatrix} \delta X \\ \delta Y \\ \delta Z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{AGD66} + \begin{bmatrix} -134 \\ -48 \\ +149 \end{bmatrix}$$

$$\begin{aligned}
X &= -4131991.9379 \text{ m} \\
Y &= +2896693.9218 \text{ m} \\
Z &= -3887822.3157 \text{ m}
\end{aligned}$$

WGS84 Ellipsoid parameters: $a = 6378137$ m $f = 1/298.257223563$

WGS84 Geodetic coordinates:

$$\begin{aligned}
\tan\phi &= \frac{Z + ve^2\sin\phi}{r} \\
\tan\lambda &= \frac{Y}{X} \\
h &= \frac{r}{\cos\phi} - \nu \\
r &= \sqrt{X^2 + Y^2}
\end{aligned}$$

solution for ϕ by iteration: $$\nu = \frac{a}{\left(1 - e^2 \sin^2 \phi\right)^{1/2}}$$

$$e^2 = f(2-f) = 6.69437999014 \times 10^{-3}$$

Iteration n	ϕ_n	ν	ϕ_{n+1}
1	$-37^\circ\, 48'\, 00.0000''$	6386171.9561 m	$-37^\circ\, 47'\, 54.5522''$
2	$-37^\circ\, 47'\, 54.5522''$	.4079	$-37^\circ\, 47'\, 54.5294''$
3	$-37^\circ\, 47'\, 54.5294''$	.4056	$-37^\circ\, 47'\, 54.5293''$
4	$-37^\circ\, 47'\, 54.5293''$	.4056	$-37^\circ\, 47'\, 54.5293''$

$$\phi = -37^\circ\, 47'\, 54.5293''$$

$$\lambda = +144^\circ\, 58'\, 04.7508''$$

$$h = 46.382 \text{ m}$$

APPENDIX 2

STANDARD MOLODENSKY TRANSFORMATION EXAMPLE

Australian Geodetic Datum 1966 (AGD66)

TO

World Geodetic System 1984 (WGS84)

Formulae:

$$\begin{aligned}\delta\phi &= \frac{1}{\rho + h}\Big\{-\delta X \sin\phi\cos\lambda - \delta Y \sin\phi\sin\lambda + \delta Z\cos\phi \\ &\quad + \frac{\nu e^2 \sin\phi\cos\phi}{a}\delta a \\ &\quad + \sin\phi\cos\phi\left(\frac{\rho}{1-f} + \nu(1-f)\right)\delta f\Big\}\end{aligned}$$

$$\delta\lambda = \frac{1}{(\nu + h)\cos\phi}(-\delta X\sin\lambda + \delta Y\cos\lambda)$$

$$\delta h = \delta X\cos\phi\cos\lambda + \delta Y\cos\phi\sin\lambda + \delta Z\sin\phi - \frac{a}{\nu}\delta a + \nu(1-f)\sin^2\phi\,\delta f$$

AGD66 Ellipsoid parameters: $a = 6378160$ m $f = 1/298.25$

WGS84 Ellipsoid parameters: $a = 6378137$ m $f = 1/298.257223563$

δ values = WGS84 – AGD66

$\delta X = -134$ m

$\delta Y = -48$ m

$\delta Z = +149$ m

$\delta a = -23$ m

$\delta f = -8.120449 \times 10^{-8}$

AGD66 Geodetic coordinates: $\phi = -37^\circ\,48'\,00.0000''$ $\lambda = +144^\circ\,58'\,00.0000''$ $h = 50.000$ m

$$\nu = \frac{a}{\left(1 - e^2\sin^2\phi\right)^{1/2}} = 6386195.179722 \text{ m}$$

$$\rho = \frac{a\left(1-e^2\right)}{\left(1 - e^2\sin^2\phi\right)^{3/2}} = 6359435.481976 \text{ m}$$

Geodetic coordinate δ values:

$$
\begin{aligned}
\delta\phi &= \frac{1}{\rho+h}\{67.249167380 - 16.888371512 + 117.733096844 \\
&\qquad +0.074662477 \\
&\qquad +0.501242257\} \\
&= \frac{168.669797446}{\rho+h} \\
&= 2.652255\times 10^{-5} \text{ radians} \\
&= 5.470669''
\end{aligned}
$$

$$
\begin{aligned}
\delta\lambda &= \frac{1}{(\nu+h)\cos\phi}\{76.923088948 + 39.303274197\} \\
&= 2.303280\times 10^{-5} \text{ radians} \\
&= 4.750856''
\end{aligned}
$$

$$
\begin{aligned}
\delta h &= 86.697104181 - 21.772357361 - 91.323150994 \\
&\quad -(-22.971061152) \\
&\quad +(-0.194156922) \\
&= -3.621500 \text{ m}
\end{aligned}
$$

Transformed Geodetic coordinates: WGS $=$ AGD $+\ \delta$ values

$$
\begin{aligned}
\phi &= -37^\circ\, 48'\, 00.0000'' + 5.4707'' &&= -37^\circ\, 47'\, 54.5293'' \\
\lambda &= +144^\circ\, 58'\, 00.0000'' + 4.7509'' &&= +144^\circ\, 58'\, 04.7509'' \\
h &= 50.000 \text{ m} - 3.622 \text{ m} &&= 46.378 \text{ m}
\end{aligned}
$$

APPENDIX 3

ABRIDGED MOLODENSKY TRANSFORMATION EXAMPLE

Australian Geodetic Datum 1966 (AGD66)

TO

World Geodetic System 1984 (WGS84)

Formulae:

$$\delta\phi = \frac{1}{\rho}\{-\delta X \sin\phi\cos\lambda - \delta Y \sin\phi \sin\lambda + \delta Z\cos\phi + \left(f\,\delta a + a\,\delta f\right)\sin 2\phi\}$$

$$\delta\lambda = \frac{1}{\nu\cos\phi}(-\delta X\sin\lambda + \delta Y\cos\lambda)$$

$$\delta h = \delta X\cos\phi\cos\lambda + \delta Y\cos\phi\sin\lambda + \delta Z\sin\phi - \delta a + \left(f\,\delta a + a\,\delta f\right)\sin^2\phi$$

AGD66 Ellipsoid parameters: $a = 6378160$ m $f = 1/298.25$

WGS84 Ellipsoid parameters: $a = 6378137$ m $f = 1/298.257223563$

δ values = WGS84 − AGD66

$\delta X = -134$ m

$\delta Y = -48$ m

$\delta Z = +149$ m

$\delta a = -23$ m

$\delta f = -8.120449 \times 10^{-8}$

AGD66 Geodetic coordinates: $\phi = -37^\circ\,48'\,00.0000''$ $\lambda = +144^\circ\,58'\,00.0000''$ $h = 50.000$ m

$$\nu = \frac{a}{\left(1 - e^2\sin^2\phi\right)^{1/2}} = 6386195.179722 \text{ m}$$

$$\rho = \frac{a\left(1-e^2\right)}{\left(1 - e^2\sin^2\phi\right)^{3/2}} = 6359435.481976 \text{ m}$$

Geodetic coordinate δ values:

$$\begin{aligned}\delta\phi &= \frac{1}{\rho}\{67.249167380 - 16.888371512 + 117.733096844 \\ &\qquad + \left(-0.077116513 + (-0.517935228)\right)\sin 2\phi\} \\ &= \frac{168.670249809}{\rho} \\ &= 2.652283 \times 10^{-5} \text{ radians} \\ &= 5.470727''\end{aligned}$$

$$\begin{aligned}\delta\lambda &= \frac{1}{\nu\cos\phi}\{76.923088948 + 39.303274197\}\\ &= 2.303298\times10^{-5}\ \text{radians}\\ &= 4.750727''\end{aligned}$$

$$\begin{aligned}\delta h &= 86.697104181 - 21.772357361 - 91.323150994\\ &\quad -(-23) + (-0.077116513 - 0.517935228)\sin^2\phi\\ &= -3.621938\ \text{m}\end{aligned}$$

Transformed Geodetic coordinates: WGS = AGD + δ values

$$\begin{aligned}\phi &= -37^\circ\,48'\,00.0000'' + 5.4707'' &&= -37^\circ\,47'\,54.5293''\\ \lambda &= +144^\circ\,58'\,00.0000'' + 4.7509'' &&= +144^\circ\,58'\,04.7509''\\ h &= 50.000\ \text{m} - 3.622\ \text{m} &&= 46.378\ \text{m}\end{aligned}$$